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WAVES IN A BED UNDER PERIODIC TANGENTIAL LOADING

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The far-zone structure of the wave field in an elastic bed on a rigid foundation is considered. The wave field is generated by a tangential periodic force applied to the bed surface. The amplitude–frequency characteristics of surface vibrations for propagating modes are found. The partition of energy between different modes is considered.

Key words: *elastic bed, periodic load, wave field.*

In applied research, it is often more important to elucidate the wave-field structure rather than to calculate displacements due to periodic loading; the key issues here are as follows: What are the modes forming the wave field? Which modes prevail among them? etc. For instance, in vibration-damping problems, one has often to know which of the modes carries the largest portion of energy under certain assumptions about the source of vibrations. Acoustic inspection problems can be considered as another example. Vibroacoustic inspection has often to be performed under conditions with possible simultaneous excitation of several propagating vibrational modes. In such cases, one has to be sure that the monitored vibrational mode is indeed excited and that the amplitude of this mode is not low compared to other excited modes. A similar situation is possible in shallow-bed explorative seismology, where the correspondence between the detected wave packet and the type of the excited wave (surface or channel one) is hard to establish.

1. Formulation of the Problem. A periodic tangential load is applied to the free surface of an elastic isotropic bed of height h resting on a rigid foundation without cohesion. We attach a cylindrical coordinate system $Or\theta z$ with axis z is directed normally to the surface inward the bed to the interface between the bed and the undeformed foundation. We reckon the angle θ from the direction of the applied tangential load. The equations of motion for an isotropic elastic continuum are [1, 2]

$$-\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + (c_p^2 - c_s^2)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + c_s^2\Delta\boldsymbol{u} = 0,$$

where $\boldsymbol{u} = (u_r, u_\theta, u_z)^{\mathrm{t}}$ is the displacement vector, $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$ are the velocities of the pressure and shear waves, λ and μ are the Lamé moduli of elasticity, and ρ is the density.

The bed surfaces ($z = 0$ and $z = h$) outside the action of the periodic load should obey the conditions of the absence of stresses on the free surface and continuity of normal displacements on the bed-foundation interface

$$\sigma_{zz}|_{z=h} = 0, \qquad \sigma_{zr}|_{z=h} = -\Pi(r,\theta)\cos\theta\,\mathrm{e}^{-i\omega t}, \qquad \sigma_{zr}|_{z=0} = 0,$$

$$\sigma_{z\theta}|_{z=h} = \Pi(r,\theta)\sin\theta\,\mathrm{e}^{-i\omega t}, \qquad \sigma_{z\theta}|_{z=0} = 0, \qquad u_z|_{z=0} = 0,$$

where σ_{zz}, σ_{zr}, and $\sigma_{z\theta}$ are the stress-tensor components:

$$\sigma_{zz} = \lambda\Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta}\Big) + 2\mu\,\frac{\partial u_z}{\partial z},$$

$$\sigma_{zr} = \mu\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big), \qquad \sigma_{z\theta} = \mu\Big(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\,\frac{\partial u_z}{\partial \theta}\Big);$$

$\Pi(r,\theta)$ is a function that describes the shear-stress distribution.

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2. Wave Field in the Far Zone. We consider the wave field generated by a point tangential load $\Pi(r) = P\delta(r)/r$, where $\delta(\boldsymbol{x})$ is the delta-function and $P = \text{const}$. To derive the expressions for the field of displacements, we use the method described in detail in [1–5]. This method consists in applying the integral Fourier–Bessel transform over the spatial variables r and θ to the equations of motion and boundary conditions with subsequent solution of the resultant algebraic problem. This yields expressions for displacements in the form of certain double integrals. To calculate the integrals, we use the contour-integration methods in the complex plane. Below, all expressions are written in the dimensionless variables

$$z' = \frac{z}{h}, \quad r' = \frac{r}{h}, \quad t' = \frac{tc_s}{h}, \quad P' = \frac{P}{\mu}, \quad \Omega = \frac{\omega h}{c_s}$$

(in what follows, the primes are omitted). Sufficiently far from the source, the dominating contribution to the wave field is made by propagating modes (i.e., by modes with real wavenumbers); this allows us to use the formula for residues at first-order poles (second-order poles appear only at certain values of Poisson's ratio ν) and write

$$u_z(r,\theta) = iP\,\mathrm{e}^{-i\Omega t}\cos\theta \sum_{n=0}^{N(\Omega)} k_n^2 \Lambda_n(\Omega) g_{zn}(\Omega, z) H_1^{(1)}(k_n r),$$

$$u_r = iP\,\mathrm{e}^{-i\Omega t}\cos\theta \sum_{n=0}^{N(\Omega)} k_n \Lambda_n(\Omega) g_{rn}(\Omega, \theta, z) H_0^{(1)}(k_n r), \tag{1}$$

$$u_\theta = iP\,\mathrm{e}^{-i\Omega t}\sin\theta \sum_{n=0}^{N(\Omega)} k_n \Lambda_n(\Omega) g_{\theta n}(\Omega, \theta, z) H_0^{(1)}(k_n r).$$

Here $N(\Omega)$ is the total number of propagating modes excited at a given frequency Ω, $H_j^{(1)}(k_n r)$ $(j = 0, 1)$ are the Hankel functions of the jth order,

$$g_{zn}(\Omega, z) = 2\eta\cosh\zeta\sinh\eta z + (\Omega^2 - 2k_n^2)\cosh\eta\sinh\ \zeta z/\zeta,$$

$$g_{rn}(\Omega,\theta,z) = 2k_n^2\cosh\zeta\cosh\eta z - (2k_n^2 - \Omega^2)\cosh\eta\cosh\zeta z + 4k_n^2\Omega^{-2}\zeta\sinh^{-1}\zeta g_{zn}(\Omega,1)\sin^2\theta\cosh\zeta z,$$

$$g_{\theta n}(\Omega,\theta,z) = [(2k^2 - \Omega^2)\cosh\eta + k^2\Omega^{-2}\zeta\sinh^{-1}\zeta g_{zn}(\Omega,1)\cos 2\theta]\cosh\zeta z,$$

$$\Lambda_n = \zeta/(2S'(k_n,\Omega)), \qquad \eta = \sqrt{k_n^2 - \Omega^2 c_s^2 c_p^{-2}}\,, \qquad \zeta = \sqrt{k_n^2 - \Omega^2}\,,$$

$$S'(k,\Omega) = 8(2k^2 - \Omega^2)k\sinh\zeta\cosh\eta + (2k^2 - \Omega^2)^2 k\Big[\frac{\cosh\zeta}{\zeta}\cosh\eta + \frac{\sinh\eta}{\eta}\sinh\zeta\Big]$$

$$- \cosh\zeta\sinh\eta(8k\zeta\eta + 4k^3\eta/\zeta + 4k^3\zeta/\eta) - 4k^3(\eta\sinh\zeta\sinh\eta + \zeta\cosh\zeta\cosh\eta),$$

and k_n is the (real) dimensionless wavenumber of the nth mode, which satisfies the dispersion equation for symmetric normal Lamb waves in an elastic bed $S(k_n,\Omega) = 0$, where $S = (2k^2 - \Omega^2)^2\sinh\zeta\cosh\eta - 4k^2\zeta\eta\cosh\zeta\sinh\eta$.

3. Normal Displacements of the Bed Surface. Figure 1 shows the dependences $A_n(\Omega) = k_n^2\Lambda_n(\Omega)g_{zn}(\Omega,1)$ with $\nu = 0.3$ for the first five modes $(n = 0,\ldots,4)$. In the case of a concentrated tangential load, the vertical displacements are primarily produced by the lowest propagating mode. The amplitude of surface vibrations excited by propagation of any other mode is smaller than one third of the amplitude of the lowest mode. Nevertheless, with allowance for the finiteness of the area to which the tangential load applied, the contribution of the lowest mode to the surface displacements at all frequencies is no longer dominating. In the case of a periodic tangential load $\Pi(r) = PH(a-r)/(\pi a^2)$ [$H(r)$ is the Heaviside function] uniformly distributed over a circular region of radius a/h, the functions $A_n(\Omega)$ should be multiplied by the factor $2hJ_1(ka/h)/(ka)$. Thus, the expression for the normal displacements of the bed surface is

$$u_z(r,\theta) = iP\,\mathrm{e}^{-i\Omega t}\cos\theta \sum_{n=0}^{N(\Omega)} A_n^a(\Omega) H_1^{(1)}(k_n r),$$

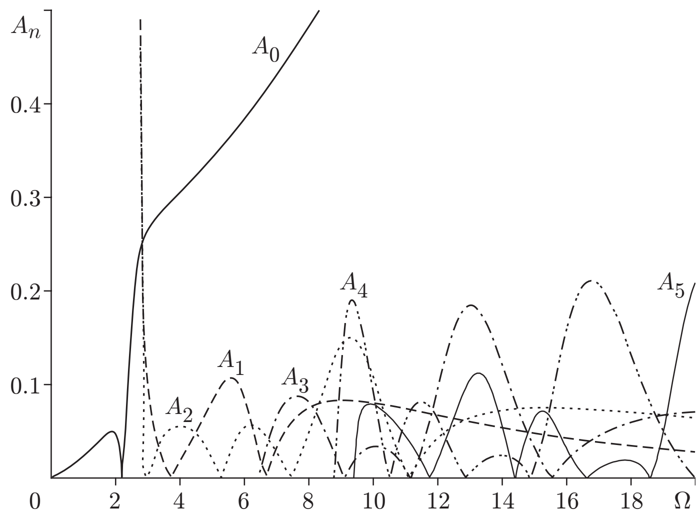

Fig. 1

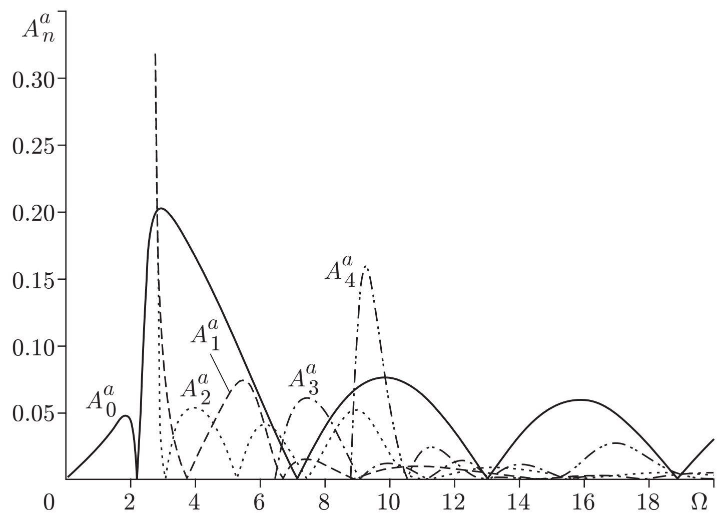

Fig. 2

where $A_n^a(\Omega) = 2ha^{-1}k_n\Lambda_n(\Omega)g_{zn}(\Omega,1)J_1(k_nah^{-1})$. The dependences $A_n^a(\Omega)$ $(n = 0,\ldots,4)$ for $a/h = 0.5$ and $\nu = 0.3$ are plotted in Fig. 2. It is seen that there exist frequencies at which the vertical displacements of the bed surface are conditioned by the highest modes only.

It is expedient to compare the dependences of the normal displacements on the tangential and vertical point loads. The expression for the normal displacements due to the concentrated vertical periodic force is well known (see [5]):

$$u_z(r) = iP\,\mathrm{e}^{-i\Omega t}\sum_{n=0}^{N(\Omega)} A_n^N H_0^{(1)}(k_nr).$$

Here $A_n^N(\Omega) = \pi\Omega^2 k_n\eta\sinh\eta\sinh\zeta/S'(k_n,\Omega)$. Figure 3 shows the dependences $A_n^N(\Omega)$ for the first three modes $(n = 0,1,2;\ \nu = 0.3)$. In contrast to the case of a vertical load, the amplitudes of normal displacements caused by tangential loading depend on the angle θ, have lower values, and display a more intricate dependence on the frequency Ω.

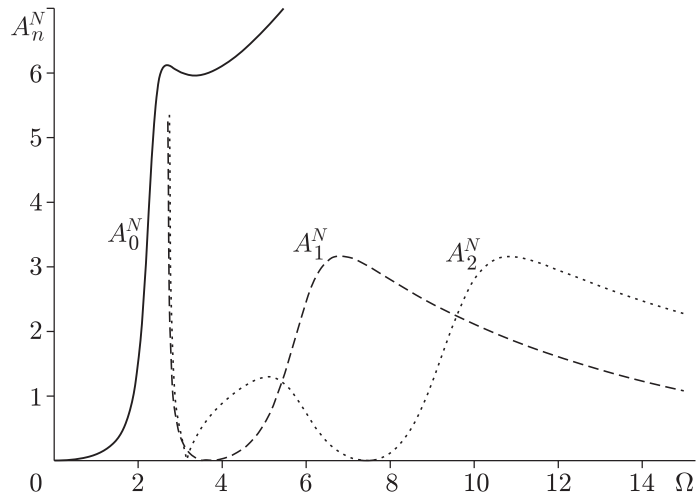

Fig. 3

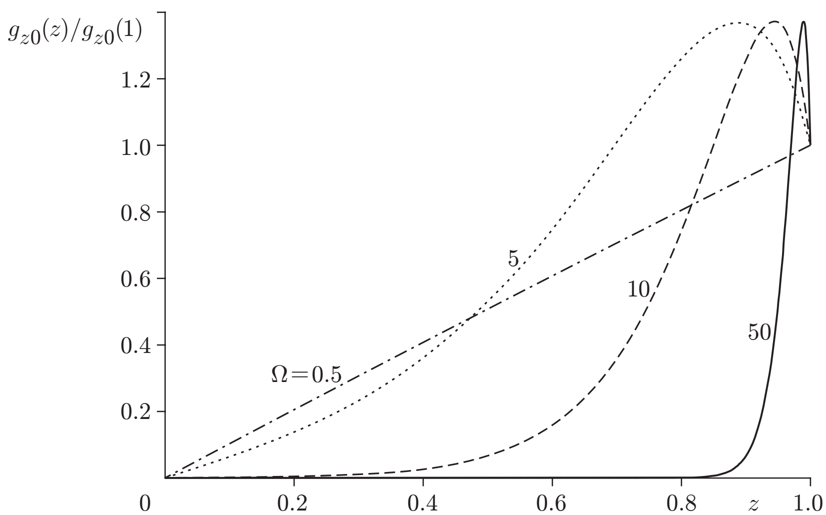

Fig. 4

Using asymptotic expansions for the Hankel functions, which are valid in the far zone for all frequencies except for values of Ω close to the barrier frequencies $\Omega_{lj} = \pi l(c_p/c_s)^j - (\pi c_p/(2c_s)+1)^j + 1$ $(l = 1, 2, \ldots, j = 0, 1)$, one can bring the first expression in (1) to the form

$$u_z(r,\theta) \approx \frac{\sqrt{2}P\cos\theta}{\sqrt{\pi r}} \sum_{n=0}^{N(\Omega)} k_n^{-1/2} A_n(\Omega) \exp\Big[-i\Big(\Omega t - k_n r + \frac{\pi}{4}\Big)\Big].$$

4. Energy Transferred by Symmetric Modes. The partition of energy between the waves of different types in an elastic half-space was considered in [6–12]. For a number of vibration-damping problems and vibration-based inspection problems, it is of interest to consider the partition of energy between the symmetric normal modes in an elastic bed. In contrast to the lowest mode, whose energy with increasing frequency is rapidly localized near the surface, modes with higher numbers display no localization of energy. Figure 4 shows the structure of vertical displacements over the bed height for the lowest mode at the frequencies $\Omega = 0.5$, 5, 10, and 50. The distributions of z-displacements for modes with numbers $n > 1$ are oscillating functions with the total number of oscillations increasing with the mode number n. Figure 5 shows the variation of the amplitude of vertical displacements across

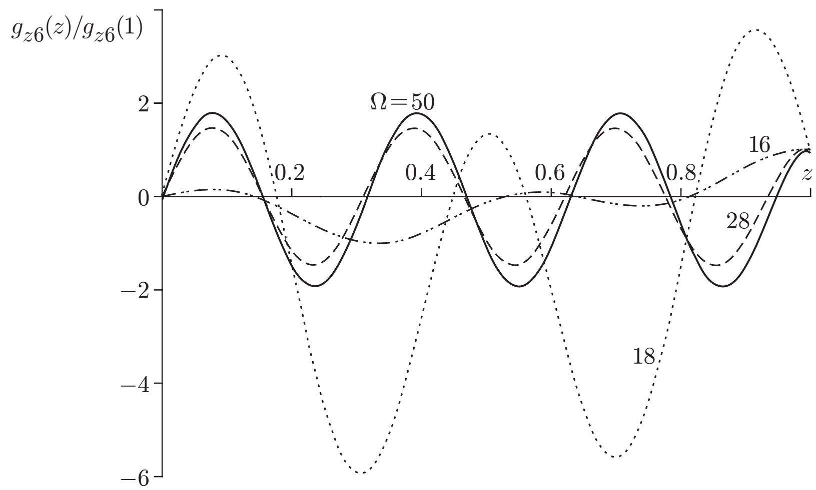

Fig. 5

the bed for the mode with $n = 6$. At certain frequencies, the amplitude of particle displacements inside the bed becomes higher than at the free surface.

The energy flux across the surface F can be calculated as

$$\iint\limits_F \left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\sigma_n}\right) dS,$$

where $\boldsymbol{\sigma_n} = \sigma_{ij} n_i$ is the stress vector on the elemental region of F with the normal $\boldsymbol{n} = (n_1, n_2, n_3)$. We represent the displacements $\boldsymbol{u}$ and the stress vector $\boldsymbol{\sigma_n}$ as

$$\boldsymbol{u}' = \hat{\boldsymbol{u}}\, \mathrm{e}^{-i\Omega t}, \qquad \boldsymbol{\sigma}'_{\boldsymbol{n}} = \hat{\boldsymbol{\sigma}}_{\boldsymbol{n}}\, \mathrm{e}^{-i\Omega t},$$

where $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\sigma}}_{\boldsymbol{n}}$ are the complex amplitudes (the complex-conjugated terms are omitted). For the dimensionless energy flux $E' = E/(\rho c_s^3 h^2)$ across the surface F, averaged over the vibration period $2\pi/\Omega$, we have

$$E' = -\frac{\Omega}{2} \operatorname{Im} \iint\limits_F (\hat{\boldsymbol{u}}, \hat{\boldsymbol{\sigma}}^*_{\boldsymbol{n}})\, dS,$$

where the asterisk indicates complex conjugation.

We consider the energy transferred by the waves across a cylindrical surface of a large radius R. The height of the cylinder is equal to the height of the bed. In view of additivity of the averaged energy flux in the bed [13], the flux E can be represented as the sum of the energies transferred by individual modes:

$$E = -\frac{\Omega}{2} \operatorname{Im} \sum_{m=0}^{N(\Omega)} \int_0^1 dz \int_0^{2\pi} d\theta (\hat{u}_r \hat{\sigma}^*_{rr} + \hat{u}_\theta \hat{\sigma}^*_{r\theta} + \hat{u}_z \hat{\sigma}^*_{rz})$$

(the subscript m denoting the mode number is omitted at the displacements and stresses). From expressions (1), one can derive expressions for $\hat{u}_r$, $\hat{u}_\theta$, and $\hat{u}_z$:

$$\begin{aligned} \hat{u}_z &= iPU_z H_1^{(1)}(k_m r), \qquad & U_z &= \cos\theta k_m^2 \Lambda_m(\Omega) g_{zm}(\Omega, z), \\ \hat{u}_r &= iPU_r H_0^{(1)}(k_m r), \qquad & U_r &= \cos\theta k_m \Lambda_m(\Omega) g_{rm}(\Omega, \theta, z), \\ \hat{u}_\theta &= iPU_\theta H_0^{(1)}(k_m r), \qquad & U_\theta &= \sin\theta k_m \Lambda_m(\Omega) g_{\theta m}(\Omega, \theta, z). \end{aligned} \tag{2}$$

Substituting (2) into the expressions relating the stresses with the displacements and taking into account that U_z, U_r, and U_θ are real-valued functions for propagating modes and that the relations

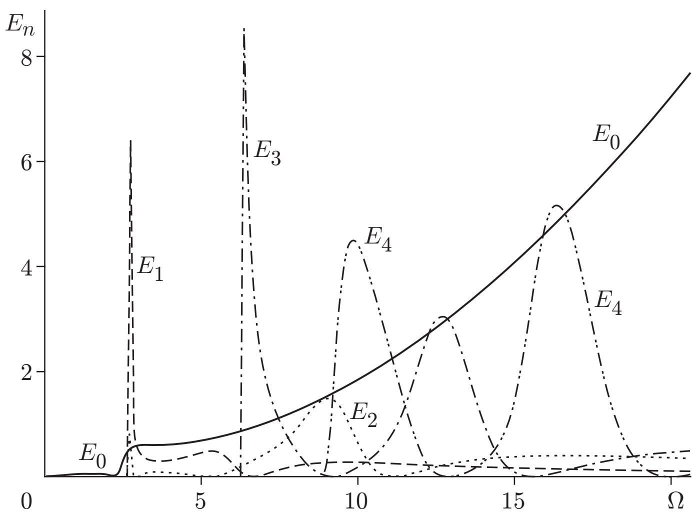

Fig. 6

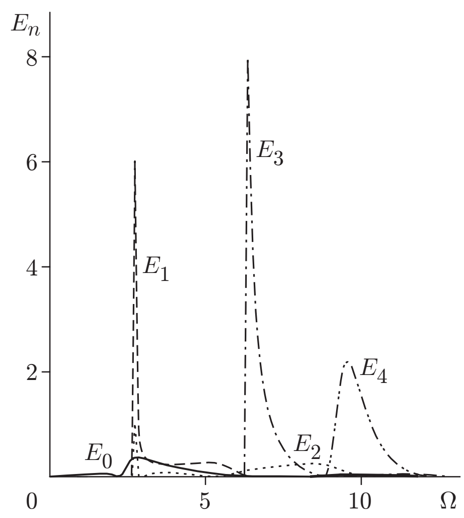

Fig. 7

$$\frac{\partial}{\partial r} H_0^{(j)}(k_m r) = -k_m H_1^{(j)}(k_m r), \qquad \frac{\partial}{\partial r} H_1^{(j)}(k_m r) \approx k_m H_0^{(j)}(k_m r) \qquad (j = 1, 2),$$

$$H_j^{(1)}(k_m r) H_l^{(2)}(k_m r) \approx \frac{2(i)^{l-j}}{\pi k_m r} \qquad (j, l = 0, 1)$$

are valid in the far zone, we obtain the expression for the energy transferred by the mth mode:

$$E_m = \frac{\Omega P^2}{\pi} \int_0^1 dz \int_0^{2\pi} d\theta \Big(\frac{2 - 2\nu}{1 - 2\nu} U_r^2 + U_\theta^2 + U_z^2 - \frac{2\nu}{1 - 2\nu} \frac{1}{k_m} U_r \frac{\partial U_z}{\partial z} + \frac{1}{k_m} U_z \frac{\partial U_r}{\partial z} \Big).$$

The partition of energy between different generated modes is shown in Figs. 6 and 7. The curves in Figs. 6 and 7 refer to a concentrated source and to a source uniformly distributed over the region $a/h = 0.5$ (Poisson's ratio is $\nu = 0.3$).

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